Electricity and Magnetism, Exam 3, 02/04/2020

5 questions, 50 points

Write your name and student number on each answer sheet. Use of a calculator is allowed. You may make use of the book and the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\mathbf{A}$ is a vector, $\hat{\boldsymbol{x}}$ is the unit vector in the x-direction, and T is a scalar. In your handwritten answers, please indicate vectors (unit vectors) with an arrow (hat) above the symbol.

1. Consider a long straight rod of radius R carrying a constant current I of uniform current density. The material of the rod has a very small susceptibility, so you can use $\chi_m = 0$.

   (a) (3 points) Calculate the magnetic field everywhere, both inside and outside the wire. Show how you obtain your result.
   Answer: Use Ampere's law: $\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}}$ *to find* $\boldsymbol{B}_{\text{outside}} = \frac{\mu_0 I}{2\pi s}\boldsymbol{\Phi}$ *and* $\boldsymbol{B}_{\text{inside}} = \frac{\mu_0 I s}{2\pi R^2}\boldsymbol{\Phi}$.

   (b) (3 points) Make a qualitative graph (that has the correct shape, but don't worry about the absolute value) of the magnetic field strength as a function of radial position. Make sure to include both the inside and outside of the wire.
   Answer: see the figure.

   (c) (3 points) Gadolinium, a silvery white metal, has a magnetic susceptibility of $\chi_m = 4.8 \times 10^{-1}$. How is the magnetic field inside and outside the wire changed when using gadolinium compared to a material with $\chi_m = 0$? Calculate this (give the expression, and show how you obtain it), and draw a separate curve in the graph that you made for question (b) to

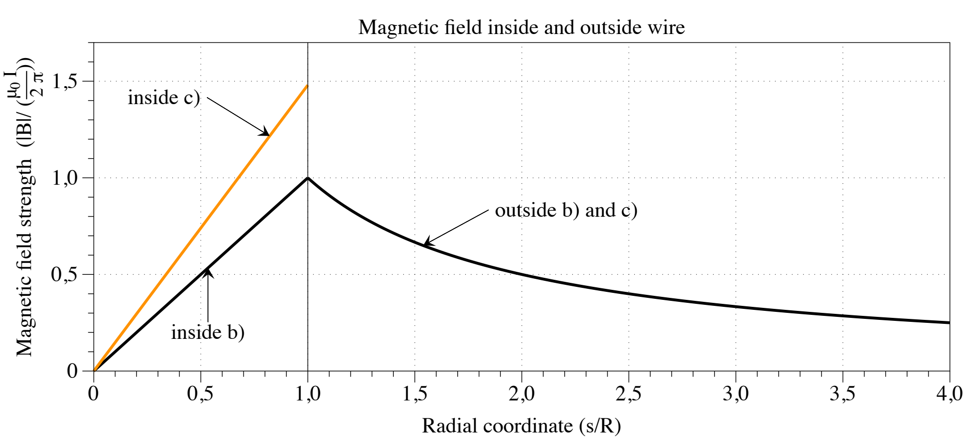


Figure 1: The graph for question 1b) and c)

show the qualitative behaviour.

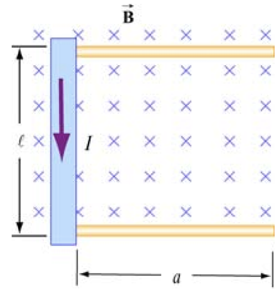
Answer: We have a linear medium, $\boldsymbol{M} = \chi_m \boldsymbol{H}$ *so* $\boldsymbol{B} = \mu_0(1+\chi_m)\boldsymbol{H}$*. We can use* $\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I_{f_{\text{enclosed}}}$ *to find, just like problem 6.17,* $\boldsymbol{B} = \frac{\mu_0(1+\chi_m)Is}{2\pi R^2}\hat{\boldsymbol{\Phi}}$ *for* $s < R$*, and* $\boldsymbol{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\Phi}}$ *for* $s > R$*.*

(d) (3 points) Calculate the bound volume current and bound surface current for this situation. Make sure to include the direction of these currents.

Answer: Like problem 6.17, but χ_m *positive instead of negative. Also see example 6.2. The direction of the bound currents is reversed compared to that example. The bound volume current* $\boldsymbol{J}_b = \chi_m \boldsymbol{J}_f$*, and* $J_f = \frac{I}{\pi s^2}$*, so* $\boldsymbol{J}_b = \frac{\chi_m \boldsymbol{I}}{\pi a^2}$ *(in the same direction as I). The bound surface current* $\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}} = \chi_m \boldsymbol{H} \times \hat{\boldsymbol{n}}$ *so that* $\boldsymbol{K}_b = -\frac{\chi_m \boldsymbol{I}}{2\pi R}$*, anti-parallel to* $\boldsymbol{I}$*.*

2. Now we will investigate the force experienced by a section of a rod with negligible χ_m in an external magnetic field.

We place a section of the rod with mass m and radius R on two parallel rails of length a. The rails are separated by a distance l, as shown in the top-view figure on the right. The rod is oriented horizontally, and initially the rod is at rest. The rod carries a current I and it can roll without slipping over the rails which are placed in a uniform magnetic field $\boldsymbol{B}$ directed into the page.



(a) (3 points) Calculate the magnetic force acting on the rod.

2 points for correct magnitude, 1 point for direction. *Answer: If we define the direction to the right as $\hat{\boldsymbol{x}}$, then $\boldsymbol{F} = \int \boldsymbol{I} \times \boldsymbol{B} dl = IBl\hat{\boldsymbol{x}}$.*
2 points for correct magnitude, 1 point for direction.

(b) (4 points) The total kinetic energy E_k

$$E_k = \frac{1}{2}mv^2 + \frac{1}{2}I_{rod}\omega^2$$

is equal to the total work done on the rod, where $I_{rod} = mR^2/2$ is the moment of inertia of the rod, and $\omega = v/R$ is its angular velocity. Calculate the final velocity of the rod, obtained after rolling distance a over the rails.

2 points for work, 1 point for equating with kinetic energy, 1 point for correct v. Answer: From the integral of the force we can find the total work: $W = \int \boldsymbol{F} \cdot d\boldsymbol{l} = Fa = IlBa$. The final velocity follows from the equation of this work to the kinetic energy, from which we find that $v = \sqrt{\frac{4IlBa}{3m}}$
2 points for work, 1 point for equating with kinetic energy, 1 point for correct v.

(c) (4 points) Now let's consider a square current loop, with current I and sides l. Find the expression for the magnetic field in the center of this current loop. Compare this to the magnetic field in the center of a circular current loop (current I) with diameter l. Which one is larger?

1 point for one side ($B_1 = \frac{\mu_0 I}{\sqrt{2}\pi l}$), 1 point for magnitude and direction square loop ($\vec{B}_{sq} = \frac{2\sqrt{2}\mu_0 I}{\pi l}\hat{z}$, if $\hat{z}$ is the direction pointing up when the loop is lying down and current is flowing counter clockwise), 1 point for circular loop ($B_c = \frac{\mu_0 I}{L}$), 1 point for correct comparison (both in terms of L and correct conclusion that circular loop field is larger). Answer: variation on problem 5.36.
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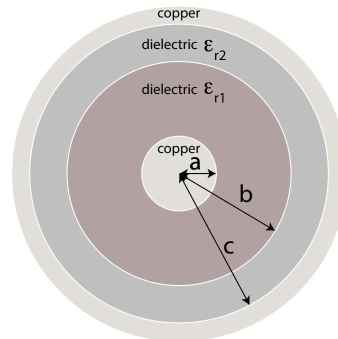
3. (10 points) A certain coaxial cable consists of a copper rod, radius a, surrounded by a concentric copper tube of inner radius c. The space between is partially filled (from a out to b) with material of dielectric constant ϵ_{r1}, and partially filled (from b out to c) by material of dielectric constant ϵ_{r2}, as shown in the figure. Find the capacitance per unit length of this cable.



Answer: Use the displacement and the fact that we are dealing with a linear dielectric to calculate the electric field inside the dielectric materials. Then calculate the potential difference between the two copper parts using the integral of the electric field. Then compute the capacitance per unit length, using the definition of the capacitance. $\oint \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{f,enc}$ *(2pt),* $\boldsymbol{D} = \frac{\lambda}{2\pi s}\hat{s}$ *(1pt) (same for* $\lambda = Q/L$*),* $\boldsymbol{D} = \epsilon_0\epsilon_r\boldsymbol{E}$ *for both regions separately (3pt),* $V = -\int_c^a \boldsymbol{E} \cdot d\boldsymbol{l}$ *(1pt),* C *per length* $= \frac{\lambda}{V}$ *(1pt),* C *per length* $= \frac{2\pi\epsilon_0}{\frac{1}{\epsilon_{r1}}\ln\frac{b}{a} + \frac{1}{\epsilon_{r2}}\ln\frac{c}{b}}$ (2pt). For mixing ϵs (-1 pt)

4. Consider three current-carrying neutral wires that lie along the three axes of a Cartesian coordinate system. This means they are all perpendicular to each other. Currents $\boldsymbol{I}_x$, $\boldsymbol{I}_y$ and $\boldsymbol{I}_z$ are flowing from $-\infty$ to $+\infty$ through these three wires in the positive $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ direction, respectively.

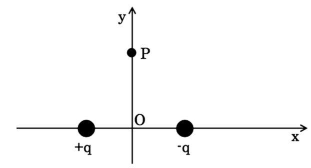
   (a) (3 points) What is the magnetic field $\boldsymbol{B}$ (magnitude and direction) at $(x, y, z) = (1, 1, 0)$?
   Answer: Use Biot-Savart and the principle of superposition. From a right-hand rule analysis of the magnetic fields, we see that the direction of the B-field from the x- and y-axes is opposite in direction (+z from x, -z from y). The magnitude depends on the difference in the currents: $\boldsymbol{B}_{xy} = \frac{\mu_0}{2\pi}(I_x - I_y)\hat{\boldsymbol{z}}$*. In addition we have the magnetic field generated by the z-axis:* $\boldsymbol{B}_z = \frac{\mu_0 I_z}{2\pi\sqrt{2}}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I_z}{4\pi}(\hat{\boldsymbol{y}} - \hat{\boldsymbol{x}})$*. The sum of* $\boldsymbol{B}_{xy}$ *and* $\boldsymbol{B}_z$ *is the total magnetic field.*

   (b) (3 points) Consider the field at a large distance from the origin. As we double the distance to the origin along the diagonal line where $x = y = z$, by which factor is the magnetic field reduced?
   Answer: Each of the wires creates a field that drops off as $1/s$ *with the distance to that wire. When doubling the distance from the origin along the diagonal, the distance to each of the wires also doubles. Therefore each of the contributions to the magnetic field, and therefore the total magnetic field also, is halved in strength.*

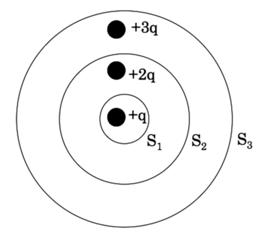
   (c) (3 points) What, if any, is the force on the other two wires caused by the magnetic field of the wire running along the x-axis? Make a sketch where you point out the relative magnitude (no need for a calculation) and the direction.
   Answer: There is a torque on the wires that run along the y- and z-axis, which is strongest close to the origin of the coordinate system. The wires at positive z- and y-values experience a force in the positive x-direction, while for negative z- and y-values the force is in the negative x-direction.

5. Conceptual questions
   (a) (2 points) A small positive charge, $+q$, is brought near but does not make contact with a small metal sphere, as shown in the figure. The metal sphere is electrically neutral (no excess charge). What is direction of the force (if any) on the $+q$ charge?

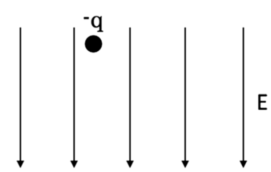
   (b) (2 points) Two small charges $+q$ and $-q$ are located along the x-axis at points that are equidistant from the origin, as shown in the figure. What is the electric field direction (if any) at point P located on the y-axis?



   (c) (2 points) Three small positive charges $(+q, +2q, +3q)$ are enclosed by three surfaces (S_1, S_2, S_3), as shown in the figure. The net electric flux through S_1 is Φ_E. What is the net electric flux through S_2, in units of Φ_E?



   (d) (2 points) In the figure, a small negatively charged particle, $-q$, is released from rest in a region containing a uniform electric field. The electric field, E, is directed downward. What is the direction of the force (if any) exerted on the negatively charged particle as it is released from rest in the electric field region?



(a) The charge will polarize the metal sphere. Since the negative charge on the sphere is closer, the effective force will be attractive. That is, $+q$ will experience a force towards the sphere.
(b) Both charges create an equal but opposite field whose vertical components cancel and horizontal components add up at P. So the field will be in $+\hat{x}$.
(c) $Q_{enc,S_2} = +1q + 2q = 3Q_{enc,S_1}$ so $\Phi_{E,S_2} = 3\Phi_E$
(d) $\vec{F} = q\vec{E}$, since q is negative $\vec{F}$ will point opposite to $\vec{E}$. So the force points upwards.

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